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Lifting Properties and Smoothness*

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INTRODUCTION

Conventions: All rings are commutative. Homomorphisms between local rings are assumed to be local, i.e., to carry the maximal ideal of one into the maximal ideal of the other.

A map of rings $\gamma: R \to S$ is (formally) *smooth* if for every surjection of artin rings $A \to B$ and commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}" near start] & A \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] \arrow[ur, phantom, "\mathrm{II}" near end] & B \end{tikzcd}$$

a lifting $S \to A$ exists such that triangles I and II commute. We are interested in weakening this condition by restricting the map $A \to B$ to be of the form $K[t]/t^{n+1} \to K[t]/t^n$ and by demanding that triangle I commute but that triangle II only commute modulo $t^{n-\delta}$ for some fixed $\delta \geqslant 0$ independent of n. If $\gamma: R \to S$ satisfies this weaker condition, we say that γ is (K, δ)-*smooth*.

Unfortunately, being (K, δ)-smooth does not imply that a map is smooth. One should keep the following two examples in mind: An inseparable field extension $k \to K$ is $(K, 0)$-smooth but not smooth; also $\mathbb{Z}/p^r \to \mathbb{Z}/p$ is $(\mathbb{Z}/p, 0)$-smooth but not smooth. In the first case the problem is due to the inseparability, and in the second case the problem is due to the map not being flat.

The basic theorem of this note is Theorem 1.9: Suppose $\gamma: R \to S$ is a flat, finitely presented, local homomorphism of complete local rings. Suppose the residue field extension K/k is separable. Then γ is smooth if and only if γ is (K, δ)-smooth. One can immediately (Corollary 1.10, Remark 1.11) globalize this statement to morphisms of schemes, algebraic spaces, etc.

This theorem can be applied to determine the smoothness of a map $\gamma: R \to S$ which is the map on minimal versal objects induced from a map $\phi: \mathcal{O}\!\ell \to \mathcal{B}$

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of functors or, more generally, of cofibered groupoids [1] (defined over C_Λ, a category of artin rings with residue field k). We make the obvious definition of (k, δ)-smoothness for ϕ, then prove: If $\mathscr{B}$ is homogeneous then ϕ is (k,δ)-smooth if and only if γ is (k, δ)-smooth (cf. Corollary 2.5). This gives a criterion for smoothness when flatness is a priori present, e.g., $\mathfrak{A}$ over k is smooth if $\mathfrak{A} \to h_k$ is (k, δ)-smooth (Corollary 2.8).

For rings, or more generally for a map of rings in a topos, (k, δ)-smoothness implies $(k, 0)$-smoothness (Proposition 1.2). It would seem reasonable that this implication would continue to hold for an arbitrary map of cofibered groupoids $\mathfrak{A} \to \mathscr{B}$, i.e., in the nongeometric case; however, for the methods here to give this would require either a notion of "extensions" for such objects (so that the proof of Proposition 1.2 would go through) or a factorization theorem for artinian rings which is stronger than the one here (Proposition 1.3). On the one hand, the theory in [2] seems to extend at best to a relatively representable situation (cf. Remark 1.11). On the other hand, simple examples show that an arbitrary map of artin rings cannot be factored as in (Proposition 1.3). Not having this implication complicates the proofs in Section 2 slightly.

1. Lifting Properties

Definition 1.1. Let K be a ring and $\gamma: R \to S$ a ring homomorphism. Let $\delta \geqslant 0$ be an integer. We say that γ is *(K, δ)-smooth* if for every commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}\quad\mathrm{II}"] & K[t]/t^{n+1} \arrow[d] \\ S \arrow[ur, dashed] \arrow[r] & K[t]/t^n \end{tikzcd}$$

there exists a map $S \to K[t]/t^{n+1}$ such that triangle I commutes and triangle II commutes modulo $t^{n-\delta}$, i.e., the two possible maps $S \to K[t]/t^{n-\delta}$ are equal.

Proposition 1.2. *(K, δ)-smooth implies $(K, 0)$-smooth.*

Proof. Let the notation be as in Definition 1.1. Adjoin to the square in Definition 1.1 another square

$$\begin{tikzcd} R \arrow[r] \arrow[d] & K[t]/t^{n+1} \arrow[r] \arrow[d] & K[t]/t^{rn+1} \arrow[d] \\ S \arrow[r] & K[t]/t^n \arrow[r] & K[t]/t^{rn} \end{tikzcd} \tag{1.2.1}$$

where the new horizontal arrows are defined by $t \mapsto t^r$. One easily sees that the right-hand square is cartesian, i.e., is a fiber product; thus to find a lifting

$S \to K[t]/t^{n+1}$ in the original square is equivalent to finding a lifting $S \to K[t]/t^{rn+1}$ in the total diagram. This being the case, in any given lifting problem (Definition 1.1), we may clearly assume n is large. In particular, we may assume that $K[t]/t^{n+1} \to K[t]/t^{n-\delta}$ is a square zero extension. Thus the lifting problem (Definition 1.1) has a solution (with triangles I and II commutative) if and only if the fiber product extension of R algebras,

$$0 \to (t^n) \to S \underset{K[t]/t^n}{\times} K[t]/t^{n+1} \to S \to 0,$$

is trivial, or equivalently that

$$0 \to (t^{rn}) \to S \underset{K[t]/t^{rn}}{\times} K[t]/t^{rn+1} \to S \to 0 \tag{1.2.2}$$

is trivial. Since γ is (K, δ)-smooth there is a map $S \to K[t]/t^{rn+1}$ which makes the total diagram (1.2.1) commute module $t^{rn-\delta}$. This means that the extension

$$0 \to (t^{rn-\delta}) \to S \underset{K[t]/t^{rn-\delta}}{\times} K[t]/t^{rn+1} \to S \to 0, \tag{1.2.3}$$

is trivial. Now it is easy to see that the extension (1.2.3) is induced from the extension (1.2.2) by "push-out" along the inclusion $(t^{rn}) \to (t^{rn-\delta})$. In this context, this inclusion should be viewed as an inclusion of S-modules where S acts through the composition $S \to K[t]/t^n \to K[t]/t^{rn}$. Taking $r > \delta$, we have that (t^{rn}) is a direct summand of $(t^{rn-\delta})$; indeed we have the decomposition $(t^{rn-\delta}) = \langle t^{rn-\delta}\rangle \oplus \cdots \oplus \langle t^{rn-1}\rangle \oplus \langle t^{rn}\rangle$ which is a decomposition as S modules because S acts through $K[t^r]/t^{rn}$ and $t^r \cdot t^{rn-i} \equiv t^{rn+r-i} \equiv 0 \bmod(t^{rn+1})$ for all $i = 1,..., \delta$. Thus, pushing-out (1.2.3) along the projection $(t^{rn-\delta}) \to (t^{rn})$ gives us back (1.2.2) which must be trivial because (1.2.3) is trivial.

PROPOSITION 1.3. *Let R be a ring. Then*

$$R[x_1 ,..., x_r]/(x_1 ,..., x_r)^{n+1} \to R[x_1 ,..., x_r]/(x_1 ,..., x_r)^n$$

can be factored into a sequence of maps $B \to C$ where there are integers s and maps such that

$$\begin{array}{ccc} B & \longrightarrow & R[t]/t^{s+1} \\ \downarrow & & \downarrow \\ C & \longrightarrow & R[t]/t^s \end{array}$$

is cartesian.

Proof. By induction on r; for $r = 1$ the lemma is trivial. Assume it for r and prove it for $r + 1$: Group variables x_i into a single variable x and a vector of variables $y = (y_1 ,..., y_r)$.

Order the monomials lexicographically to obtain a list

$$x^n,\ x^{n-1}y,\ x^{n-2}y^2,\dots,\ xy^{n-1},\ y^n,$$

where y^k denotes the lexicographically ordered list of monomials in the variables $y_1,\dots,y_r$ of degree k.

Now it is clear that $x \mapsto t$ and $y \mapsto 0$ gives a cartesian square

$$\begin{tikzcd}
R[x, y]/(x^{n+1}, x^{n-1}y,\dots, xy^{n-1}, y^n) \arrow[r] \arrow[d] & R[t]/t^{n+1} \arrow[d] \\
R[x, y]/(x^n, x^{n-1}y,\dots, xy^{n-1}, y^n) \arrow[r] & R[t]/t^n.
\end{tikzcd}$$

Thus we search to factor

$$\begin{tikzcd}
R[x, y]/(x^{n+1}, x^n y^1,\dots, x^{n+2-k}y^{k-1}, x^{n+1-k}y^k, x^{n-k-1}y^{k+1},\dots, xy^{n-1}, y^n, y^{n+1}) \arrow[d] \\
R[x, y]/(x^{n+1}, x^n y,\dots, x^{n+2-k}y^{k-1}, x^{n-k}y^k, x^{n-k-1}y^{k+1},\dots, xy^{n-1}, y^n, y^{n+1}),
\end{tikzcd}$$

by maps of the form $x \mapsto t^p$, $y_i \mapsto t^{q_i}$. Now denote the components of y^k by $(y_1{}^k, y_2{}^k,\dots)$ then we have to find a map $x \mapsto t^p$, $y_i \mapsto t^{q_i}$ so that

$$\begin{tikzcd}
R[x, y]/(\dots, x^{n+1-k}y^k_{b-1}, x^{n+1-k}y_b{}^k, x^{n-k}y^k_{b+1},\dots,) \arrow[r] \arrow[d] & R[t]/t^{N+1} \arrow[d] \\
R[x, y]/(\dots, x^{n+1-k}y^k_{b-1}, x^{n-k}y_b{}^k, x^{n-k}y^k_{b+1},\dots,) \arrow[r] & R[t]/t^N,
\end{tikzcd}$$

is cartesian. Note that if $y_b{}^k = y_1^{b_1} \cdot \ldots \cdot y_r^{b_r}$ then $N = (n-k)p + \sum b_i q_i$. Reading left to right in these lists of monomials we obtain the following necessary and sufficient conditions for this map to exist:

1. $(k+1-h)p + q_{i_1} + \cdots + q_{i_h} > \sum b_i q_i$ for $h = 0,\dots, k-1$
2. $p + \sum a_i q_i > \sum b_i q_i$ if $a < b$
3. $\sum c_i q_i > \sum b_i q_i$ if $b < c$
4. $q_{i_1} + \cdots + q_{i_h} > (h-k)p + \sum b_i q_i$ for $h = k+1,\dots, n+1$.

We make some observations on these inequalities. Setting $h = k+1$ in 4 we see that $q_i > p$ is a necessary condition. The inequalities are homogeneous; thus a (positive) rational solution implies a (positive) integral solution. They are even open conditions so that a real solution would imply a rational one; however, we will not need this.

Accordingly, we set $p = 1$ and $q_i = 1 + \delta_i$ and these conditions become

$$\begin{array}{lrl}
1'. & 1 + \delta_{i_1} + \cdots + \delta_{i_h} > \sum b_i\delta_i & \text{for} \quad h = 0,\ldots, k-1, \\
2'. & 1 + \sum a_i\delta_i > \sum b_i\delta_i & \text{if} \quad a < b, \\
3'. & \sum c_i\delta_i > \sum b_i\delta_i & \text{if} \quad b < c, \\
4'. & \delta_{i_1} + \cdots + \delta_{i_h} > \sum b_i\delta_i & \text{for} \quad h = k+1,\ldots, n.
\end{array}$$

Now for any numbers $\delta_1, \ldots, \delta_r$ such that

$$\frac{1}{k} - \frac{1}{rk} < \delta_i < \frac{1}{k}, \tag{1.4}$$

one sees immediately that conditions $1'$, $2'$, and $4'$ hold. Thus there remains only to find a solution to $3'$ subject to (1.4). But $3'$ is homogeneous, and since $\sum c_i = \sum b_i = k$ it is translation invariant. It follows that any (possibly negative) solution yields, by appropriate scaling, one satisfying (1.4). We can proceed by induction on r: noting the lexicographic order of the monomials, we can choose $\delta_2, \ldots, \delta_r$ so that

$$\sum_{i\geqslant 2} c_i\delta_i > \sum_{i\geqslant 2} b_i\delta_i, \tag{1.5}$$

holds for those c with $c_1 = b_1$. For the remaining c, $c_1 < b_1$, so δ_1 can be chosen large enough *negatively* so that

$$\sum_{i\geqslant 1} c_i\delta_i > \sum_{i\geqslant 1} b_i\delta_i,$$

holds in all cases.

COROLLARY 1.6. *$R \to S$ is (K, δ)-smooth if and only if for every commutative square*

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & K[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^{n+N} \arrow[d] \\
S \arrow[r] \arrow[ur, dashed] & K[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^{n},
\end{tikzcd}$$

the dashed arrow exists to commute. The above statement with $K[[x_1, \ldots, x_r]]$ replacing $K[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^{n+N}$ also holds.

PROPOSITION 1.7. *Let $A = K[[x_1, \ldots, x_n]]/(f_1, \ldots, f_m)$ where $f(0) = 0$ and K is a field. Let $k \to K$ be a ring homomorphism such that A/k is (K, δ)-smooth. Then A/K is smooth.*

Proof. By Proposition 1.2 we may assume $\delta = 0$. First, let us show that A/K is $(K, 0)$-smooth: Given a commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}\quad\mathrm{II}"] & K[t]/t^{q+1} \arrow[d] \\ A \arrow[r] \arrow[ur, dashed, "\phi"] & K[t]/t^q, \end{tikzcd}$$

the indicated lifting ϕ exists such that triangle II commutes, but where ϕ is only k-linear. Since the total square commutes,

$$K \longrightarrow A \xrightarrow{\phi} K[t]/t^{q+1} \longrightarrow K,$$

must be the identity, and hence ϕ must in fact be K-linear, i.e., triangle I also commutes.

If $A \cong K[[x_1, \ldots, x_n]]$, i.e., $m = 0$, then A/K is smooth. We proceed by induction on m. Suppose for the moment some $(\partial f_i/\partial x_j)_0$, say $(\partial f_m/\partial x_n)_0$, is a unit, i.e., is nonzero (K is a field). Expanding f_m in its Taylor expansion and using $f(0) = 0$ gives

$$f_m(x) = \left(\frac{\partial f_m}{\partial x_1}\right)_0 x_1 + \cdots + \left(\frac{\partial f_m}{\partial x_n}\right)_0 x_n + g(x),$$

where $(\ )_0$ denotes the value after setting $x = 0$ and where $g(x) \equiv 0 \bmod (x)^2$. Now $(\partial f_m/\partial x_n)_0$ is a unit; thus we can solve $f_m(x) = 0$ for x_n, so that A will be presented as

$$K[[x_1, \ldots, x_{n-1}]]/(g_1, \ldots, g_{m-1}).$$

Thus by induction A/K would be smooth. Henceforth we can (and do) assume $(\partial f_i/\partial x_j)_0 = 0$. This means that $A/m_A^2 \cong K[[x_1, \ldots, x_n]]/(x)^2$. By Corollary 1.6 this isomorphism can be lifted to a map $\psi: A \to K[[x_1, \ldots, x_n]]$ in

$$\begin{tikzcd} K \arrow[r] \arrow[d] & K[[x_1, \ldots, x_n]] \arrow[d] \\ A \arrow[r] \arrow[ur, dashed, "\psi"] & K[[x_1, \ldots, x_n]]/(x)^2 \end{tikzcd}$$

Since ψ induces an isomorphism on cotangent spaces which is the inverse to the isomorphism induced by $\pi: K[[x_1, \ldots, x_n]] \to A$, $\pi\psi$ and $\psi\pi$ are both surjective, hence bijective; hence π and ψ are both isomorphisms. Thus A/K is smooth.

Corollary 1.8. *Suppose A/k is a local ring complete and separated for the m_A-adic topology whose residue field K is a separable field extension of k. Suppose m_A/m_A^2 is finite-dimensional over K. Then A is noetherian, and A/k is (K, δ)-smooth if and only if A/k is smooth.*

Proof. Since K/k is separable we may assume we have a factorization $k \to K \to A \to K$. Since $m_A/m_A{}^2$ is finite-dimensional over K, say $x_1, \ldots, x_n \in A$ induces a basis, $K[[X_1, \ldots, X_n]] \to A$ by $X_i \mapsto x_i$ is surjective, and A is noetherian. Thus assuming A/k is (K, δ)-smooth, the hypotheses of Proposition 1.7 are satisfied so A/K is smooth. Since K/k is separable, K/k is smooth; hence, A/k is smooth. The converse is obvious.

THEOREM 1.9. *Suppose $R \to S$ is a flat, finitely presented, local homomorphism of complete local rings. Suppose the residue field extension K/k is separable. Then S/R is (K, δ)-smooth if and only if S/R is smooth.*

Proof. It suffices to show $A = S \otimes k$ is smooth over k. By (1.2.3) it is (K, δ)-smooth and $m_A/m_A{}^2$ is finite-dimensional over K, since m_S is finitely generated. Thus we are done by the previous corollary.

COROLLARY 1.10. *Let $f: X \to Y$ be a flat morphism of schemes (or algebraic spaces) locally of finite type over an algebraically closed field k. Suppose there is a $\delta \geqslant 0$ such that for any $\alpha \in X(k[t]/t^n)$ there is a $\beta \in X(k[t]/t^{n+1})$ whose image in $X(k[t]/t^{n-\delta})$ is the same as that of α. Then f is smooth.*

Proof. Follows from Theorem 1.9.

Remark 1.11. One can vary the theme of Corollary 1.10 a little bit. What one has to control is the flatness, the finiteness, and the separability of the residue field extensions; then one can apply Theorem 1.9. For example, assume $f: X \to Y$ is flat, locally of finite presentation, and for all $x \in X$ assume $k(x)/k(f(x))$ is separable and $\mathcal{O}_{(f)x}{}^\wedge \to \mathcal{O}_x{}^\wedge$ is $(k(x), \delta_x)$ smooth. Then f is smooth.

On a more functorial level, one can consider a map of functors $f: X \to Y$ where the functors X and Y are, say, from (schemes)° to (sets). When f is *relatively representable*, i.e., when for every representable V and map $V \to Y$ the fiber product $U = V \times_Y X$ is representable, one can put conditions on f by putting conditions on the maps $U \to V$ which represent F, e.g., flatness, finiteness, etc. Smoothness in this case obviously coincides with the usual notion of smoothness for such functors. Thus one obtains a corollary identical to Corollary 1.10 for relatively representable morphisms f of functors (schemes)° $\to$ (sets). These ideas are extended in the next section to groupoids cofibered over a category of artin rings.

2. CRITERIA FOR SMOOTH FUNCTORS

In this section we apply the results of the previous section to maps of functors, or more generally, allowing our functors to be groupoid valued instead of set valued, to maps of cofibered groupoids. For brevity we use the notation and terminology of Rim [3]: Λ denotes a local complete noetherian ring with residue field k, C_Λ denotes the category of local artinian Λ-algebras which are quotients of $\Lambda[[x_1,..., x_n]]$ for some n. All of our functors and cofibered groupoids $\mathcal{O}\!\mathcal{l}$ are defined over C_Λ ; $\mathcal{O}\!\mathcal{l}$ has a natural extension $\hat{\mathcal{O}\!\mathcal{l}}$ over the category $\hat{C}_\Lambda$ of all rings which are quotients of $\Lambda[[x_1,..., x_n]]$ for some n.

For comparison with (k, δ)-smoothness which will be defined shortly, recall that a map $\phi: \mathcal{O}\!\mathcal{l} \to \mathcal{B}$ of cofibered groupoids over C_Λ is *smooth* if for every surjection $A' \to A$ in C, the map

$$\mathcal{O}\!\mathcal{l}(A') \to \mathcal{O}\!\mathcal{l}(A) \overset{2}{\underset{\mathcal{B}(A)}{\times}} \mathcal{B}(A')$$

is essentially surjective. This is equivalent to saying for every $\pi: b' \to \phi(a)$ in $\mathcal{B}$ which lies over the surjection $A' \to A$ there are maps $\alpha: a' \to a$ and $\beta: \phi(a') \to b'$ such that $\pi\beta = \phi(\alpha)$, i.e.,

$$\begin{tikzcd} \phi(a') \arrow[r, "\beta"] \arrow[dr, "\phi(\alpha)"'] & b' \arrow[d, "\pi"] \\ & \phi(a) \end{tikzcd}$$

commutes.

DEFINITION 2.1. Let $\delta \geqslant 0$ be an integer. Write k_m for $k[t]/t^m$. A map $\phi: \mathcal{O}\!\mathcal{l} \to \mathcal{B}$ of cofibered groupoids over C_Λ is *(k, δ)-smooth* if the image of an object under

$$\mathcal{O}\!\mathcal{l}(k_n) \overset{2}{\underset{\mathcal{B}(k_n)}{\times}} \mathcal{B}(k_{n+N}) \to \mathcal{O}\!\mathcal{l}(k_{n-\delta}) \overset{2}{\underset{\mathcal{B}(k_{n-\delta})}{\times}} \mathcal{B}(k_{n+N})$$

is isomorphic to the image of an object from $\mathcal{O}\!\mathcal{l}(k_{n+N})$.

One may take $N = 1$ in the definition; moreover, the definition is equivalent to saying for every $\pi: b' \to \phi(a)$ in $\mathcal{B}$ which lies over $k[t]/t^{n+N} \to k[t]/t^n$ there are maps $\alpha: a' \to \bar{a}$ and $\beta: \phi(a') \to b'$ such that $\bar{a} =$ image of a in $\mathcal{O}\!\mathcal{l}(k[t]/t^{n-\delta})$ and

$$\begin{tikzcd} \phi(a') \arrow[r, "\beta"] \arrow[ddr, "\phi(\alpha)"'] & b' \arrow[d, "\pi"] \\ & \phi(a) \arrow[d] \\ & \phi(\bar{a}) \end{tikzcd}$$

commutes.

Of course $h_S \to h_R$ is smooth, respectively (k, δ)-smooth, if and only if $R \to S$ is such, and a groupoid $\mathcal{O}\!\mathcal{l}$ is *smooth* (respectively *(k, δ)-smooth*) if $\mathcal{O}\!\mathcal{l} \to h_\Lambda$ is smooth (respectively (k, δ)-smooth). Here, for $R \in \hat{C}_\Lambda$, we write h_R for the discrete groupoid $h_R(A) = \mathrm{Hom}_\Lambda(R, A)$.

PROPOSITION 2.2.

1. *$\phi\colon \mathcal{A} \to \mathcal{B}$ is (k, δ_1)-smooth and $\psi\colon \mathcal{B} \to \mathcal{C}$ is (k, δ_2)-smooth then $\psi\phi\colon \mathcal{A} \to \mathcal{C}$ is $(k, \delta_1 + \delta_2)$-smooth.*

2. *If $\phi\colon \mathcal{A} \to \mathcal{B}$ is (k, δ)-smooth then so is $\mathcal{A} \times^2_{\mathcal{B}} \mathcal{C} \to \mathcal{C}$ for any map $\mathcal{C} \to \mathcal{B}$.*

3. *If the composition $\mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C}$ is (k, δ)-smooth then*

   (a) *ϕ essentially surjective implies ψ is (k, δ)-smooth.*

   (b) *If $\mathcal{B}$ is semihomogeneous, $\mathcal{C}$ is homogeneous, and $|\psi|(k[\epsilon])\colon |\mathcal{B}(k[\epsilon])| \to |\mathcal{C}(k[\epsilon])|$ is injective then ϕ is (k, δ)-smooth.*

Before giving the proof of Proposition 2.2 we remark that the statements obtained from by removing the prefix "(k, δ)" are all true. (cf. [3]). One should be warned however that if $\phi\colon \mathcal{A} \to \mathcal{B}$ is smooth then ϕ is essentially surjective; hence in Proposition 2.2, 3b for smoothness one can also conclude that ψ is smooth by 3a for smoothness. This is false for (k, δ)-smoothness. For example, the map $h_S \to h_R$ given by the $(k, 0)$-smooth ring homomorphism $R = \mathbb{Z}/p^2 \to S = \mathbb{Z}/p$ is not essentially surjective.

Proof of (2.2). The proofs of statements 1, 2, and 3a are completely straightforward and will be omitted. The proof of 3b is similar to that in the smooth case [3, 2.9]: Let $b' \to \phi(a)$ lie over $k[t]/t^{n+1} \to k[t]/t^n$ and let $\bar{a}$ be the image of a in $\mathcal{A}(k[t]/t^{n-\delta})$.

Since $\psi\phi$ is (k, δ)-smooth there are maps $a' \to \bar{a}$ and $\psi\phi(a') \to \psi(b')$ (which by applying direct image we may assume to lie over $k[t]/t^{n+1} \to k[t]/t^{n-\delta}$ and $id_{k[t]/t^{n+1}}$, respectively) such that

$$\begin{tikzcd} \psi\phi(a') \arrow[r] \arrow[dr] & \psi(b') \arrow[d] \\ & \psi\phi(\bar{a}) \end{tikzcd}$$

commutes.

Setting $b = \phi(a')$, $b_0 = \phi(\bar{a})$ we are done by the following lemma.

LEMMA 2.3. *Let $\mathcal{B} \to^{\psi} \mathcal{C}$ be given. Suppose $\mathcal{B}$ is semihomogeneous, $\mathcal{C}$ is homogeneous, and the tangent map $|\mathcal{B}(k[\epsilon])| \to |\mathcal{C}(k[\epsilon])|$ is injective. Then a diagram*

$$\begin{tikzcd} b \arrow[r, dashed] \arrow[dr] & b' \arrow[d] \\ & b_0 \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} A' \arrow[r, "id"] \arrow[dr, "f"'] & A' \arrow[d, "f"] \\ & A \end{tikzcd}$$

where $f\colon A' \to A$ is surjective can be completed if it can be completed after applying ψ.

Proof. We may assume the extension $f\colon A' \to A$ is small with kernel ϵ. Since $\mathscr{B}$ is semihomogeneous there is a commutative diagram

$$\begin{tikzcd} & b'' \arrow[dl] \arrow[dr] & \\ b \arrow[dr] & & b' \arrow[dl] \\ & b_0 & \end{tikzcd} \quad \text{above} \quad \begin{tikzcd} & A' \times_A A' \arrow[dl] \arrow[dr] & \\ A' \arrow[dr] & & A'. \arrow[dl] \\ & A & \end{tikzcd}$$

Since $\mathscr{C}$ is homogeneous $\psi(b'') = \psi(b) \times_{\psi(b_0)} \psi(b')$ so by [3, 2.8(2)] the image of $\psi(b'')$ in $|\mathscr{C}(k[\epsilon])|$ by the canonical map $A' \times_A A' \to k[\epsilon]$ is zero. Since $|\mathscr{B}(k[\epsilon])| \to |\mathscr{C}(k[\epsilon])|$ is injective the image of b'' in $|\mathscr{B}(k[\epsilon])|$ is zero and the lemma follows from [3, 1.5(2)].

Next recall that if $\mathscr{A}$ is a cofibred groupoid, a formal object $\alpha \in \hat{\mathscr{A}}$ over $S \in \hat{C}_\Lambda$ is *versal* for $\mathscr{A}$ if the induced map $\alpha^{\#}\colon h_S \to \mathscr{A}$ (which sends a homomorphism $f\colon S \to A$ to $f_*\alpha \in \mathscr{A}(A)$) is smooth. If moreover $\alpha^{\#}$ induces a bijection of tangent spaces $\mathrm{Hom}_\Lambda(S, k[\epsilon]) \to |\mathscr{A}(k[\epsilon])|$ then α is *minimally versal.*

Now suppose that we have a map $\phi\colon \mathscr{A} \to \mathscr{B}$ where both $\mathscr{A}$ and $\mathscr{B}$ admit versal formal objects α/S and β/R, respectively. Since β is versal there is a map $\beta \to \hat{\phi}(\alpha)$ in $\hat{\mathscr{B}}$ which induces a commutative diagram:

$$\begin{tikzcd} h_S \arrow[r] \arrow[d, "\alpha^{\#}"'] & h_R \arrow[d, "\beta^{\#}"] \\ \mathscr{A} \arrow[r, "\phi"'] & \mathscr{B}. \end{tikzcd} \tag{2.4}$$

One has immediately from Proposition 2.2 the following:

COROLLARY 2.5. *In the diagram* (2.4)

1. *If S/R is (k, δ)-smooth then ϕ is (k, δ) smooth.*

2. *If ϕ is (k, δ)-smooth, β is minimal, and $\mathscr{B}$ is homogeneous, then S/R is (k, δ) smooth.*

COROLLARY 2.6. *If α/S is versal for $\mathscr{A}$ then $\mathscr{A}$ is (k, δ)-smooth if and only if S/Λ is (k, δ)-smooth.*

Remark 2.7. The statements about smoothness corresponding to those of Corollaries 2.5 and 2.6 are all valid; however, the one corresponding to 2 of Corollary 2.5 is true without the hypothesis that $\mathscr{B}$ is homogeneous. Indeed, since β is minimal versal there is a cosection $S \to R$ which is the identity on R. It follows that there is a map $T = R[[x_1, \ldots, x_n]] \to S$ which induces an isomorphism and consequently on $T/(m_T{}^2 + m_\Lambda) \to S/(m_S{}^2 + m_\Lambda)$. Now

$\phi\alpha^{\#}$ smooth says $\hat{\phi}(\alpha)$ is versal for $\mathscr{B}$; thus there is a map $S \to T$ which lifts $S \to T/(m_T{}^2 + m_A)$ and induces an isomorphism on cotangent spaces. Hence $S \to T$ and $T \to S$ are isomorphisms, and S/R is smooth. This fact, pointed out to me by Artin, seems to have been missed in [3]. I do not know if the homogeneity assumption in 2 of Corollary 2.5 is necessary.

COROLLARY 2.8. *Suppose $\Lambda = k$ and α/S is versal for $\mathfrak{A}$; then the following are equivalent:*

1. *$\mathfrak{A}$ is (k, δ)-smooth.*
2. *$\mathfrak{A}$ is smooth.*
3. *S/k is (k, δ)-smooth.*
4. *S/k is smooth.*

Proof. Combine Corollary 2.6 with Remark 1.11.

Remark 2.9. As in Remark 1.11 the obstruction to (k, δ)-smoothness for maps of cofibered groupoids is the flatness. For example, in (2.5.2) if one knew a priori that S/R was flat then one could conclude S/R (and hence ϕ) was smooth. But even if the map $\phi\colon \mathfrak{A} \to \mathscr{B}$ is relatively representable by flat maps, the map $h_S \to h_R$ (or equivantly $R \to S$) in (2.4) is not necessarily flat. This occurs for the map $\phi\colon \mathrm{Def}_X \to \mathrm{Def}_Y$ between the groupoids of deformations of a rational double point Y (defined by $z^2 = xy$) and its minimal resolution X. ϕ is relatively representable by flat maps, but if $\mathrm{char}(k) = 2$ the map on the minimal versal objects is not flat. The point here is that the square (2.4) is not cartesian. For a complete discussion of ϕ see [1].

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